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SOLIDIFICATION OF A LIQUID ON A MOVING SHEET

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Abstract—This paper considers the growing of a solid layer on a sheet that moves through a liquid and which is kept at a temperature below freezing. The convection in the liquid is fully taken into account. It is found that the thickness of the layer is proportional to the square root of the distance from the point where the sheet enters the body of liquid. The main difficulty lies in determining the factor of proportionality in this relationship. Asymptotic expressions are derived for this factor in the case where latent heat is much greater than sensible heat. Also presented are approximate solutions valid for very small (liquid metals) and very large (polymers) values of the Prandtl number.

NOMENCLATURE

b, $\eta_s\sigma^{1/2}$ (6.3);
c_n, expansion coefficient (4.9);
c_p, specific heat of liquid at constant pressure $[\mathrm{J \cdot kg^{-1} \cdot K^{-1}}]$;
f, function defining the velocity field (3.1);
f_n, expansion function (4.2);
F, function defined by (5.5);
k, thermal conductivity of liquid $[\mathrm{J \cdot m^{-1} \cdot s^{-1} \cdot K^{-1}}]$;
k_s, thermal conductivity of solid $[\mathrm{J \cdot m^{-1} \cdot s^{-1} \cdot K^{-1}}]$;
K, k_s/k;
L, latent heat of fusion [J/kg];
s, σ_s/σ;
T_0, temperature of ambient fluid [K];
T_f, temperature of the sheet [K];
T_s, freezing temperature [K];
u, velocity component in the x-direction [m/s];
u_s, velocity of the sheet [m/s];
v, velocity component in the y-direction [m/s];
x, coordinate measuring distance along the sheet [m];
y, coordinate measuring distance from the sheet [m].

Greek symbols

γ, ρ_s/ρ;
δ, thickness of solid layer [m];
η, similarity variable (3.3);
η_s, value of η at $y = \delta$;
$\bar{\eta}$, approximate value of η_s (4.14);
$\hat{\eta}$, $\eta - \eta_s$;
θ, normalized temperature (3.2);
θ_n, expansion function (4.3);
θ_s, value of θ at freezing;
κ, thermal diffusivity of liquid $[\mathrm{m^2/s}]$;
κ_s, thermal diffusivity of solid $[\mathrm{m^2/s}]$;
μ, transformed similarity variable (5.5);
ν, kinematic viscosity of liquid $[\mathrm{m^2/s}]$;
ρ, density of liquid $[\mathrm{kg/m^3}]$;
ρ_s, density of solid $[\mathrm{kg/m^3}]$;
σ, Prandtl number: ν/κ;
σ_s, Prandtl number: ν/κ_s;
ω, $2\gamma\sigma L/c_p/(T_0 - T_f)$;
$\bar{\omega}$, parameter defined by (4.13);
$\tilde{\omega}$, ω/σ;
Ω, $K(1-\theta_s)$.

1. INTRODUCTION

THE SOLIDIFICATION of a liquid on a wall cooled to below the freezing point has been the subject of many theoretical studies. Practical applications of this work may be found in various fields of engineering. In metallurgy, for example, we have the application of thin metallic layers on solid walls, thin wires, tubes etc. When a liquid metal is conducted through a tube or a channel, it is important to make sure that no blockage will occur through the development of a solid internal crust [1]. Similar applications, such as ice formation inside water mains, were mentioned by Zerkle *et al.* [2]. The solidification of polymeric liquids is an important process in the electrical and chemical industries. This process finds application in the casting of an insulating coating on electricity cables or wires. In semiconductor technology the growing of silicon layers etc. is of importance.

Apart from this variety of applications, there are also a variety of ways in which solid layers may be grown. The simplest way is to keep the system at rest and to let the solidification take place through pure temperature action. Some recent references are [3–6]. Another possibility is to force the liquid to flow along the wall on which the solidification is to occur. Examples are given in references [7–11]. In [12] the solidification of a liquid film on a vertical wall has been studied.

A third way to grow a layer is to let the subcooled wall move through a liquid otherwise at rest. The solidification will then occur in a boundary layer near the moving wall. In this way layers can be created continually by leading the bare wall into the liquid at a certain location and by taking it out of the system at a point where the layer has grown to the required thickness. Instead of varying in time, as was true in the first class of problems, the thickness of the layer now varies spatially. Moreover, by controlling the velocity of the wall, we are able to influence the rate at which the layer will grow in the lengthwise direction. From the point of view of process technology this method may be regarded as very attractive. It is likely that the yield of a continuous process will be much larger than that of a batch-type process.

An interesting application of this technique is described in a recent paper by Chopra *et al.* [13]. To produce a continuous lead sheet they use a rotating drum which is slightly immersed in a melt. The drum is cooled internally so that the surface will be at a temperature below the freezing point, even if it is in contact with the melt. As a result a solid layer will grow on the drum. At the upper side this layer is taken off the drum in the shape of a continuous lead sheet. "While", to cite the authors, "this process is now well established, little is known from a fundamental point of view about the solidification and heat-transfer process between the casting drum and the lead bath". This shows that there is a need for some theory to describe this kind of process.

In this paper we intend to consider a sheet which at some location enters a body of liquid. The sheet is kept at a temperature below freezing. We shall consider steady conditions only, where the solid layer has assumed its perfect state. The main object of the paper will be to determine the thickness of the layer as a function of the distance from the inlet.

From a mathematical point of view the problem considered here is related to the problem area concerned with the flow about and the cooling of continuous moving objects, such as sheets or cylinders. A few references are [14–17].

2. FORMULATION OF THE PROBLEM

We consider a continuous moving sheet which enters a semi-infinite fluid region through a slit in a bounding wall. This wall is assumed to be insulated. The heat capacity of the sheet is assumed to be large, so that it will remain at a fixed temperature T_f, which is below the temperature T_s at which change of phase will take place.

The surrounding fluid has a temperature T_0 which is larger than T_s, i.e.

$$T_f < T_s < T_0. \tag{2.1}$$

The sheet moves at a velocity u_s. Under these conditions the fluid will solidify near the moving surface and we obtain a solid layer which becomes thicker as we move further away from the slit (Fig. 1).

To describe this problem we need a coordinate system. At any point in the fluid or in the solid layer, x is the distance from the bounding wall and y is the distance above the sheet. The position of the solid–liquid interface is given by $y = \delta(x)$.

The differential equations governing this problem describe both the transport of momentum and the transport of heat. To describe the motion of the fluid we have

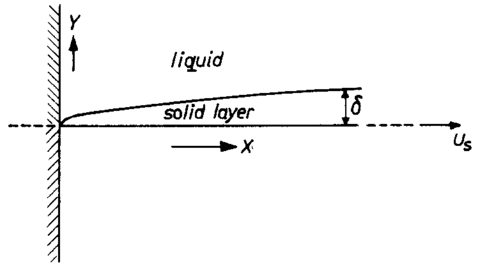

FIG. 1. Geometrical configuration.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2.3}$$

which are boundary-layer-type equations. It is well known that these equations may be used if $u_s x/\nu \gg 1$, i.e. where x is large enough. However, this condition does not seriously restrict the applicability of the results. For water, e.g. we have $\nu \sim 10^{-6}$ m^2/s, so that for $u_s \sim 0.1$ m/s the condition is $x \gg 10^{-5}$ m. Only in the case of a very slowly moving plate the results will be of limited value. For many other materials similar conditions can be found. Clearly the equations are considered in the region $y \geqslant \delta$.

Transport of heat takes place both in the fluid and in the solid layer, i.e.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} \quad \text{if} \quad y > \delta \tag{2.4}$$

$$u_s\frac{\partial T}{\partial x} = \kappa_s\frac{\partial^2 T}{\partial y^2} \quad \text{if} \quad 0 \leqslant y < \delta. \tag{2.5}$$

The boundary conditions of this problem are prescribed at $y = 0$, $y = \delta$ and $y \to \infty$. At the surface of the sheet we simply have

$$T = T_f \quad \text{at} \quad y = 0. \tag{2.6}$$

To describe the ambient conditions we put

$$u \to 0, \quad T \to T_0 \quad \text{if} \quad y \to \infty. \tag{2.7}$$

At the interface the conditions are more complicated. First we have a prescribed velocity and temperature

$$u = u_s, \quad T = T_s \quad \text{at} \quad y = \delta. \tag{2.8}$$

Next we have to demand continuity of mass transport through the interface. By considering an infinitesimal pill-box, with one side in the fluid and one in the solid, we are able to derive the following condition

$$\rho\left[u\frac{\mathrm{d}\delta}{\mathrm{d}x} - v\right]_{y=\delta+0} = \rho_s u_s\frac{\mathrm{d}\delta}{\mathrm{d}x}. \tag{2.9}$$

The thermal balance at the interface can also be given by means of the pill-box model. Applying boundary-layer approximations we easily find

$$k_s\frac{\partial T}{\partial y}\bigg|_{y=\delta-0} - k\frac{\partial T}{\partial y}\bigg|_{y=\delta+0} = \rho_s u_s L\frac{\mathrm{d}\delta}{\mathrm{d}x}. \tag{2.10}$$

3. SOLUTION

The problem can be solved by means of a similarity transformation. We shall present this transformation directly, omitting a detailed description of how it can be obtained. We then have

$$u = u_s \frac{df}{d\eta}, \quad v = \left(\frac{\nu u_s}{x}\right)^{1/2} \left(\eta \frac{df}{d\eta} - f\right) \tag{3.1}$$

$$T = T_0 - (T_0 - T_f)\theta(\eta) \tag{3.2}$$

$$\eta = y \left(\frac{u_s}{4\nu x}\right)^{1/2}. \tag{3.3}$$

It can be proved by substitution that (3.1) satisfies (2.2). For the similarity transformation to be valid it is necessary that the solid–liquid interface can be represented by a fixed value of η, that is by $\eta = \eta_s$. It then follows that

$$\delta = \eta_s \left(\frac{4\nu x}{u_s}\right)^{1/2} \tag{3.4}$$

where η_s is an unknown that we shall find later by integrating the complete set of equations and boundary conditions. This set can be found by substitution of (3.1)–(3.4) into (2.3)–(2.10). The result is

$$f''' + 2ff'' = 0 \qquad (\eta_s \leqslant \eta < \infty) \tag{3.5}$$

$$\theta'' + 2\sigma f\theta' = 0 \qquad (\eta_s < \eta < \infty) \tag{3.6}$$

$$\theta'' + 2\sigma_s \eta\theta = 0 \qquad (0 \leqslant \eta < \eta_s) \tag{3.7}$$

$$\eta = 0: \quad \theta = 1 \tag{3.8}$$

$$\eta = \eta_s \qquad f = \gamma\eta_s \qquad f' = 1 \tag{3.9}$$

$$\theta = \theta_s \tag{3.10}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=\eta_s+0} = K \left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=\eta_s-0} + \omega\eta_s \tag{3.11}$$

$$\eta \to \infty: \quad f' \to 0, \quad \theta \to 0 \tag{3.12}$$

where

$$\omega = \frac{2\gamma\sigma L}{c_p(T_0 - T_f)}, \quad K = \frac{k_s}{k}. \tag{3.13}$$

A prime stands for differentation with respect to the argument. It is clear that the temperature field and the flow field are coupled inasmuch as η_s is an unknown. In general, this will complicate the numerical integration of the present system. Another complication is the large number of parameters influencing this problem, viz. σ, σ_s, γ, K and ω. We shall show later on that matters can be simplified considerably by assuming certain parameters to be large or small. We shall succeed in finding solutions that reveal an explicit dependence upon the parameters of the problem.

The integration of (3.7) can be done immediately:

$$\theta = 1 - (1-\theta_s)\frac{\operatorname{erf}(\eta\sigma_s^{1/2})}{\operatorname{erf}(\eta_s\sigma_s^{1/2})}, \quad (0 \leqslant \eta < \eta_s). \tag{3.14}$$

The boundary condition (3.11) can now be rewritten as

$$\eta = \eta_s + 0:$$

$$\theta' = \omega\eta_s - 2K\left(\frac{\sigma_s}{\pi}\right)^{1/2}(1-\theta_s)\frac{\exp(-\eta_s^2\sigma_s)}{\operatorname{erf}(\eta_s\sigma_s^{1/2})} \tag{3.15}$$

where the derivative is understood to be taken at the fluid side of the interface. The remaining part of the paper will be devoted to the solution of the system consisting of the equations (3.5), (3.6), (3.8)–(3.10), (3.12) and (3.15).

4. THE CASE $\eta_s \ll 1$

For many materials the latent heat L predominates over the sensible heat, i.e. $L \gg c_p\Delta T$ if the temperature differences are not extreme. This means that $\omega \gg 1$ if the Prandtl number σ is not too small. In a later section we shall consider the case of small σ in more detail. For the moment we shall assume a Prandtl number of order unity. It will be of interest to determine the value of ω for a typical fluid in this Prandtl number range: water. For water we have $\sigma = 13.4$, $L = 334000$ J/kg, $\gamma = 0.92$, $c_p = 4200\,\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$, so that

$$\omega \sim \frac{1960\,\text{K}}{(T_0 - T_f)}. \tag{4.1}$$

It follows that $\omega \gg 1$ in most practical cases.

Integrating (3.6) from $\eta = \eta_s$ to $\eta = \infty$ we can easily prove that $\theta'(\eta_s) < 0$. From (3.15) we then see that $\omega \to \infty$, leaving the remaining parameters unchanged, necessarily implies $\eta_s \to 0$. In consequence, we seem justified in searching for a solution valid for small values of η_s. We also note that the equations (3.5)–(3.6) do not change under the transformation $\hat{\eta} = \eta - \eta_s$, whence we can solve the system by introducing the expansions

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\hat{\eta})(\gamma\eta_s)^n \tag{4.2}$$

$$\theta(\eta) = \theta_s \sum_{n=0}^{\infty} \theta_n(\hat{\eta})(\gamma\eta_s)^n \tag{4.3}$$

and substituting these in the requisite equations and boundary conditions. In doing so, we shall disregard the boundary condition (3.15) for the moment. The perturbation functions satisfy the equations.

$$\begin{aligned} f_0''' + 2f_0 f_0'' &= 0 \\ f_1''' + 2f_0 f_1'' + 2f_0'' f_1 &= 0 \\ f_2'' + 2f_0 f_2'' + 2f_0'' f_2 &= -2f_1 f_1'' \end{aligned} \tag{4.4}$$

$$\begin{aligned} \theta_0'' + 2\sigma f_0 \theta_0' &= 0 \\ \theta_1'' + 2\sigma(f_0\theta_1' + f_1\theta_0') &= 0 \\ \theta_2'' + 2\sigma(f_0\theta_2' + f_1\theta_1' + f_2\theta_0') &= 0 \end{aligned} \tag{4.5}$$

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and the boundary conditions are

$$\hat{\eta} = 0: \quad f_n = 0 \ (n \neq 1), \quad f_1 = 1$$
$$f_n' = 0 \ (n \neq 0), \quad f_0' = 1 \tag{4.6}$$
$$\theta_n = 0 \ (n \neq 0), \quad \theta_0 = 1 \tag{4.7}$$
$$\hat{\eta} \to \infty: \quad f_n' \to 0, \quad \theta_n \to 0. \tag{4.8}$$

The function f_0' describes the well-known Sakiadis–Howarth velocity profile [18, 19]. The equations can be integrated numerically without much difficulty. Results which are useful for the present analysis are tabulated in Table 1. Using this table we are able to find the value of the expression

$$\theta'(\eta_s) = -c_0 - c_1\eta_s - c_2\eta_s^2 + \cdots \tag{4.9}$$

where

$$c_i = -\theta_s \gamma^i \theta_i'(0). \tag{4.10}$$

Table 1. Numerical figures which are useful in the calcuation of (4.9)

σ	$-\theta_0'(0)$	$-\theta_1'(0)$	$-\theta_2'(0)$
0.1	0.145725	0.08584	0.0410
0.2	0.267358	0.19012	0.0763
0.3	0.372512	0.30407	0.1115
0.4	0.465730	0.42357	0.1485
0.6	0.627041	0.67143	0.2300
0.8	0.765217	0.92512	0.3218
1.0	0.887496	1.18148	0.4232
2.0	1.366517	2.47376	1.0510
3.0	1.731765	3.76748	1.8406
4.0	2.038364	5.05945	2.7616
6.0	2.550827	7.63865	4.9333
8.0	2.981643	10.2133	7.4799
10.0	3.360587	12.7848	10.351

Having found the formal solution, which is valid for a given but arbitrary value of $\eta_s \ll 1$, we are now in a position to determine the true value of η_s by demanding that (4.9) should be equal to (3.15).

Expanding (3.15) for small values of η_s we obtain

$$\theta'(\eta_s) = \omega\eta_s - \frac{\Omega}{\eta_s}\{1 - \tfrac{2}{3}\eta_s^2\sigma_s + O(\eta_s^4\sigma_s^2)\} \tag{4.11}$$

where

$$\Omega = K(1-\theta_s).$$

It follows that η_s should be calculated from

$$0 = -\Omega + c_0\eta_s + \bar{\omega}\eta_s^2 + c_2\eta_s^3 + \cdots \tag{4.12}$$

where

$$\bar{\omega} = \omega + c_1 + \tfrac{2}{3}\sigma_s\Omega. \tag{4.13}$$

Since we assume $\omega \gg 1$, i.e. $\bar{\omega} \gg 1$, it is clear that a first approximation can be obtained by neglecting terms of $O(\eta_s^3)$. The result is

$$\eta_s \sim \bar{\eta}_s = \frac{2\Omega/c_0}{1 + (1 + 4\Omega\bar{\omega}/c_0^2)^{1/2}}. \tag{4.14}$$

Including the term $c_2\eta_s^3$ we obtain a better approximation

$$\eta_s \sim \bar{\eta}_s\left\{1 - \frac{c_2\bar{\eta}_s^2}{(c_0^2 + 4\Omega\bar{\omega})^{1/2}}\right\}. \tag{4.15}$$

The applicability of these results is restricted by the condition $\eta_s \ll 1$, i.e. by $\bar{\eta}_s \ll 1$. Therefore, the values of $\bar{\omega}$, Ω and c_0 should be such that the R.H.S. of (4.4) is much smaller than unity.

5. THE CASE $\sigma \ll 1$

If the Prandtl number is small, the temperature boundary layer will extend far beyond the viscous boundary layer. We may therefore obtain a first approximation of the temperature field by substituting the asymptotic velocity field into (3.6). The solution then satisfying (3.9) is simply

$$\theta \sim \theta_s \exp\{-2\sigma f(\infty)(\eta - \eta_s)\}. \tag{5.1}$$

As this solution must also satisfy (3.15) we find

$$f(\infty) \sim \frac{1}{2\sigma\theta_s}\left\{2\left(\frac{\sigma_s}{\pi}\right)^{1/2}\Omega\frac{\exp(-\eta_s^2\sigma_s)}{\operatorname{erf}(\eta_s\sigma_s^{1/2})} - \tilde{\omega}\eta_s\sigma\right\} \tag{5.2}$$

where we have introduced

$$\tilde{\omega} = \omega/\sigma. \tag{5.3}$$

Since we not only have the Prandtl number σ, but also the Prandtl number σ_s based on the properties of the solid state, it will be convenient to introduce the ratio

$$s = \sigma_s/\sigma. \tag{5.4}$$

It does not seem unreasonable to assume that $\sigma \ll 1$ implies $\sigma_s \ll 1$ and that s is of order unity.

For a given value of η_s, we can also determine $f(\infty)$ by integrating equation (3.5) using the conditions (3.8), (3.9) and (3.12). Performing such an integration for various values of η_s and γ we are able to produce the graphs of Fig. 2. The value of η_s is then determined by the intersection of (5.2) and the requisite graph of Fig. 2.

If $f(\infty)$ is large, i.e. if η_s is large, Fig. 2 cannot be used. However, we are now able to do an asymptotic analysis. Indeed, if we introduce the transformation

$$f(\eta) = \eta_s F(\mu), \quad \mu = (\eta - \eta_s)\eta_s \tag{5.5}$$

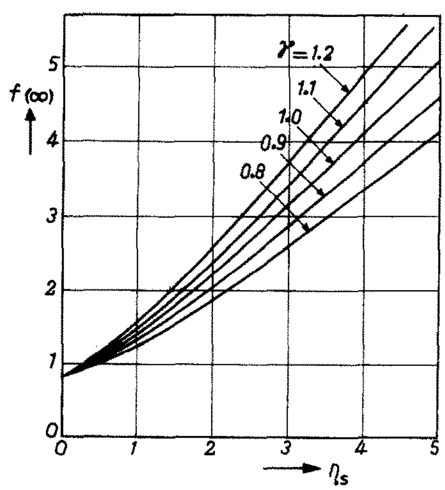

FIG. 2. The value of $f(\infty)$ as a function of η_s.

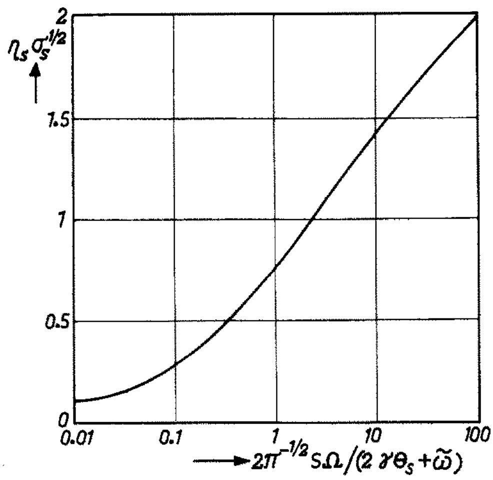

FIG. 3. symptotic behaviour of η_s valid for $\eta_s > 5$ and $\sigma \ll 1$.

the system for the function f is redefined as follows

$$F''' + 2FF'' = 0 \tag{5.6}$$

$$F(0) = \gamma, \quad F'(0) = \eta_s^{-2}, \quad F'(\infty) = 0. \tag{5.7}$$

The system admits the solution

$$F \sim \gamma + \tfrac{1}{2}\gamma^{-1}\eta_s^{-2}(1 - \mathrm{e}^{-2\gamma\mu}) + O(\eta_s^{-4}) \tag{5.8}$$

when $\eta_s \gg 1$. This yields

$$f(\infty) \sim \gamma\eta_s + (2\gamma\eta_s)^{-1} + O(\eta_s^{-3}). \tag{5.9}$$

We can now determine η_s by equating (5.2) and (5.9). Using the first term in the expansion (5.9) only we are able to derive the result of Fig. 3.

6. THE CASE $\sigma \gg 1$

In large-Prandtl-number fluids the temperature boundary layer is very much thinner than the viscous boundary layer. We may therefore derive an approximate solution for the temperature field by substituting into (3.6) the linear approximation

$$f \sim (\gamma - 1)\eta_s + \eta \tag{6.1}$$

which is valid near the solid–liquid interface. The ensuing equation can be solved analytically:

$$\theta \sim \theta_s \frac{\operatorname{erfc}\{[(\gamma-1)\eta_s + \eta]\sigma^{1/2}\}}{\operatorname{erfc}(\gamma\eta_s\sigma^{1/2})}. \tag{6.2}$$

By taking the first derivative of (6.2) with respect to η and demanding that this should be equal to (3.15) we obtain an algebraic equation for the unknown η_s. In terms of the more suitable unknown

$$b = \eta_s \sigma^{1/2} \tag{6.3}$$

this equation reads

$$\frac{\theta_s}{\exp(\gamma^2 b^2)\operatorname{erfc}(\gamma b)} + \frac{\pi^{1/2}}{2}\tilde{\omega} b = \frac{\Omega s^{1/2}}{\exp(b^2 s)\operatorname{erfc}(b s^{1/2})}. \tag{6.4}$$

In general, the value of b must be calculated numerically from (6.4). In view of the large number of parameters involved it does not seem possible to present useful tables or graphs. Of course, if b is extreme we could attempt to derive approximate results. Thus we have

$$b \sim \frac{\pi^{1/2}\Omega\theta_s^{-1}}{1 + \{1 + \pi\Omega(\tilde{\omega} + 4\pi^{-1}\theta_s\gamma + \tfrac{2}{3}s\Omega)\theta_s^{-2}\}^{1/2}} \tag{6.5}$$

if $b \ll 1$, which is the equivalent of (4.14) valid for σ large. For $b \gg 1$ (6.4) remains an implicit equation for b and the asymptotic result will be of limited usefulness.

7. CONCLUSIONS

The main object of this paper has been to determine the thickness of the solid layer, δ. By (3.4) this thickness was shown to be related to a dimensionless parameter η_s, which is a function of a great many physical constants such as σ, σ_s, γ, K and $\tilde{\omega}$. In general, this functional dependence cannot be given explicity. However, we were able to derive useful asymptotic results that seem to cover almost all practical cases within the limits given by the conditions of Section 2.

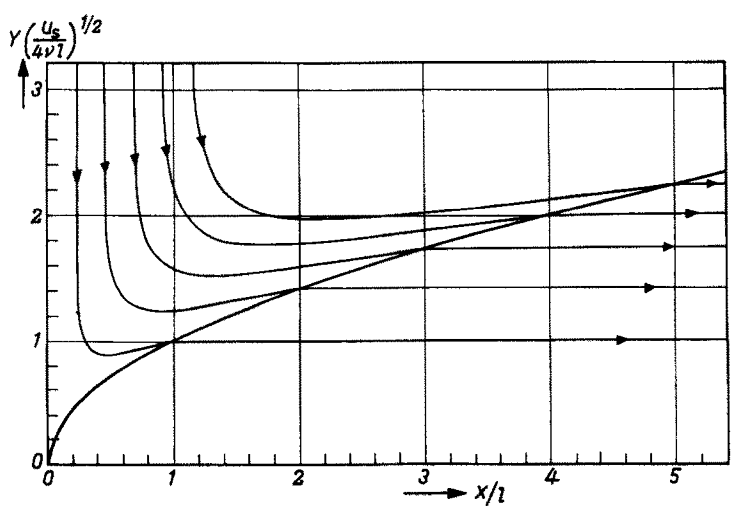

FIG. 4. Streamline behaviour before and after freezing ($\gamma = 0.5$).

It is also worthy of note that the flow field reveals some peculiar characteristics. In Fig. 4 we have presented some flow lines for $\gamma < 1$ and these show that fluid particles reach a minimum distance from the surface of the sheet before actually becoming part of the solid layer. The reason, of course, is that the material expands upon solidification. In order to demonstrate the effect clearly we have done the calculation for the rather unrealistic value $\gamma = 0.5$. When considering the graph one ought to realize that the normal coordinate has been blown up, again to show the effect clearly.

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SOLIDIFICATION D'UN LIQUIDE SUR UNE PLAQUE MOBILE

Résumé—On considère la croissance d'une couche solide sur une plaque qui se déplace dans un liquide et qui est maintenue à une température au dessous de celle de solidification. La convection dans le liquide est complètement prise en compte. On trouve que l'épaisseur de la couche est proportionnelle à la racine carrée de la distance à partir du point où la plaque pénètre dans le liquide.

Le difficulté principale est la détermination du facteur de proportionnalité. Des expressions asymptotiques sont obtenues pour ce facteur dans le cas où la chaleur latente est très grande par rapport à la chaleur sensible. On présente aussi des solutions approchées valables pour des valeurs très petites (métaux liquides) et très grandes (polymères) du nombre de Prandtl.

DAS ERSTARREN EINER FLÜSSIGKEIT AUF EINEM SICH BEWEGENDEN BAND

Zusammenfassung—Die Arbeit betrachtet das Anwachsen einer erstarrten Schicht auf einem Band, welches sich durch eine Flüssigkeit hindurch bewegt und auf einer Temperatur unterhalb der Erstarrungstemperatur gehalten wird. Dabei wird die Konvektion der Flüssigkeit mitberücksichtigt. Es zeigte sich, daß die Eisschichtdicke proportional der Quadratwurzel aus der Entfernung von der Eintauchstelle auf dem Band ist.

Die Hauptschwierigkeit liegt in der Bestimmung dieses Proportionalitätsfaktors. Für den Fall, daß die latente Wärme erheblich größer als die fühlbare Wärme ist, werden asymptotische Ausdrücke für diesen Faktor abgeleitet. Außerdem werden Näherungslösungen für sehr kleine (Flüssigmetalle) und sehr große (Polymere) Prandtl-Zahlen angegeben.

ЗАТВЕРДЕВАНИЕ ЖИДКОСТИ НА ПЕРЕМЕЩАЮЩЕЙСЯ ПЛАСТИНЕ

Аннотация — В статье рассматривается рост слоя твердого вещества на пластине, перемещающейся в жидкости и имеющей температуру ниже точки замерзания с учетом конвекции в жидкости. Найдено, что толщина слоя пропорциональна квадратному корню расстояния от точки входа пластины в жидкость. Основная трудность состояла в определении коэффициента пропорциональности в этом соотношении. Выведены асимптотические выражения для коэффициента пропорциональности в случае, когда значение скрытой теплоты намного превосходит величину теплосодержания. Приводятся также приближенные решения, справедливые для очень малых (жидкие металлы) и очень больших (полимеры) значений числа Прандтля.